2) **(30%)** Solve the following <u>nonhomogeneous</u> IBVP:

PDE: $u_t = \alpha^2 u_{xx} + \sin 3\pi x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = \sin \pi x \qquad 0 \le x \le 1$

Show all details.

$u(x,t) = \sum T_m(t) \sin m\pi x$

$T_m'(t) \sin m\pi x = \alpha^2 (-m\pi)^2 \sin m\pi x + \sin 3\pi x$

$T_m(t)$

$\left(T_m'(t) + (\alpha m\pi)^2 T_m(t)\right) \sin m\pi x = \sin 3\pi x$

for $m = 3 \qquad T_3'(t) + (3\alpha\pi)^2 T_3(t) = 1$

$m \neq 3 \qquad T_m'(t) + (m\alpha\pi)^2 T_m(t) = 0$

1st order differential equation

IC: $u(x,0) = \sin \pi x$

$\left(T_m'(0) + (\alpha m\pi)^2 T_m(0)\right) \sin m\pi x = \sin \pi x$

for $m = 1 \quad T_1'(0) + (\alpha\pi)^2 T_1(0) = 1$

$m \neq 1 \quad T_m'(0) + (\alpha m\pi)^2 T_m(0) = 0$

$m = 3 \quad T_3'(t) + (3\alpha\pi)^2 T_3(t) = 1 \Rightarrow T_3(t) = C e^{-(3\alpha\pi)^2 t} + \frac{1}{(3\alpha\pi)^2}$

$T_3(0) = 0$

$\Leftrightarrow C = -\frac{1}{(3\alpha\pi)^2}$

$\Rightarrow T_3(t) = \frac{1}{(3\alpha\pi)^2}\left(1 - e^{-(3\alpha\pi)^2 t}\right)$

3) (25%)

a) Use the Fourier transform to solve the IVP:

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a)

Let $U(\xi, t) = \mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\xi x}\, dx$

PDE

$\Rightarrow U_t(\xi,t) = -\alpha^2 \xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + (\alpha^2 \xi^2 \beta)\, U(\xi,t) = 0 \qquad$ ODE

$U_t(\xi,t) = C\, e^{-(\alpha^2\xi^2\beta)t}$

<u>IC:</u> $U(\xi,0) = C = \Phi(\cancel{t}) \qquad \left(\Phi = \mathcal{F}\,\phi(x)\right)$

$U(\xi,t) = \Phi(t) \cdot e^{-(\alpha^2 t)\xi^2} \cdot e^{-\beta t}$

$u(x,t) = \mathcal{F}^{-1}\left(U(\xi,t)(x)\right) = \mathcal{F}^{-1}\Phi(t) * \mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right)$?

$= e^{-\beta t}\phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right)$

$\mathcal{F}^{-1}\left(e^{-(\alpha^2 t)\xi^2}\right) = \mathcal{F}^{-1}\left(e^{-\xi^2/4a^2}\right) = e^{-a^2x^2} \cdot a\sqrt{2}$

$\frac{1}{4a^2} = \alpha^2 t \Rightarrow a = \frac{1}{2\alpha\sqrt{t}}$

5) **(15%)** Solve the following <u>nonlinear</u> IVP:

PDE: $u_t + uu_x = 0 \qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = x+1 \qquad -\infty < x < \infty$

Hint: a) Try a solution of the form $u(x,t) = X(x)T(t)$.

b) The general solution of the 1st order nonlinear ODE

$$T' + kT^2 = 0 \quad \text{is} \quad T(t) = \frac{1}{kt+b}$$

$u(x,t) = X(x)T(t)$

$XT' + XT \cdot X'T = 0$

$\Rightarrow \cancel{X}T' = -\cancel{X}X'T^2$

$\Rightarrow \frac{T'}{T^2} = -X' = \alpha$

$\Rightarrow T' - \alpha T^2 = 0$ and $X' = -\alpha$

let $\alpha = -K$

$\Rightarrow T' + KT^2 = 0$ and $X' = K$

$\boxed{X = Kx + a}$

(1) $\Rightarrow T(t) = \frac{1}{Kt+b}$

(2) $(X) = ?$

$\boxed{u(x,t) = \frac{Kx+a}{Kt+b} = \frac{x+A}{t+B} \Rightarrow u(x,0) = \frac{x+A}{B} = x+1 \quad A = B = 1 \Rightarrow u = \frac{x+1}{t+1}}$

$\cancel{u(x,0) = x+1}$

$\Rightarrow \frac{1}{b} = x+1 \Rightarrow b(x+1) = 1 \Rightarrow \boxed{b = \frac{1}{x+1}}$

$T(t) = \frac{1}{Kt + \frac{1}{x+1}} = \frac{1}{\frac{Kt(x+1)+1}{x+1}} = \boxed{\frac{x+1}{Kt(x+1)+1}}$